PAR LPTHE 94-34

BROWN-HET-965

August 1994

COLLECTIVE HAMILTONIANS WITH KAC-MOODY ALGEBRAIC CONDITIONS*

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ABSTRACT

We describe the general framework for constructing collective–theory Hamiltonians whose hermicity requirements imply a Kac–Moody algebra of constraints on the associated Jacobian. We give explicit examples for the algebras $sl(2)_k$ and $sl(3)_k$. The reduction to W_n–constraints, relevant to n-matrix models, is described for the Jacobians.

⋆ Work supported in part by the Department of Energy under contract DE-FG02-91ER40688 - Task A

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1. Introduction

Recent work on the collective field–theoretical approach to the description of matrix models emphasized the particular form and consistency structure of the Hamiltonian [1], suggesting the use of this structure as a guideline to the construction of Hamiltonians relevant to the formulation of n-matrix models. The proposed framework is the following:

One considers a generalized invariant collective Hamiltonian, for a matrix or vector theory, of the form:

$$\mathcal{H} = \sum_{i,j} \left(\Omega_{ij}(\phi) \frac{\partial}{\partial \phi_i} + \omega_j(\phi) \right)^\dagger \frac{\partial}{\partial \phi_j} \tag{1.1}$$

Here $\Omega_{ij}(\phi)$ is a symmetric, field–dependent kernel. This Hamiltonian is non-Hermitian. It must be conjugated by the Jacobian encapsulating the change from the original dynamical variables to particular invariant collective variables. The resulting Hamiltonian is hermitian since the conjugation re-establishes its unitarity properties [2]. Hermiticity conditions are first of all the vanishing of the linear part of the Hamiltonian. This is equivalent to the following differential equations for $J(\phi)$:

$$\left\{ \Omega_{ij}(\phi) \frac{\partial}{\partial \phi_j} + \omega_i(\phi) \right\} J = 0 \tag{1.2}$$

The symmetry of the kernel $\Omega_{ij}(\phi)$ under interchange of indices i, j is the second requirement to get a hermitian Hamiltonian. This Hamiltonian reads:

$$\begin{aligned} \tilde{H} = \frac{1}{2} \sum_{i,j} & \left(\frac{\partial}{\partial \phi_i} \Omega_{ij}(\phi) \frac{\partial}{\partial \phi_j} - \frac{1}{2} \frac{\partial^2 \Omega_{ij}}{\partial \phi_i \partial \phi_j} \right. \\ & \left. + \frac{1}{4} (\omega_i + \sum_k \frac{\partial \Omega_{ik}}{\partial \phi_k}) \Omega^{-1}_{ij} (\omega_j + \sum_k \frac{\partial \Omega_{jk}}{\partial \phi_k}) \right) - \frac{1}{4} \sum_i \frac{\partial \omega_i}{\partial \phi_i} \end{aligned} \tag{1.3}$$

The set of differential operators in (1.2) must therefore close an algebra in order for $J(\phi)$ to exist at least locally in the ϕ-space. Reciprocally, if one is given a set of

first order differential operators closing an algebra, the procedure (1.1)–(1.3) gives rise to an associated Hermitian Hamiltonian, provided the kernel $\Omega_{ij}(\phi)$ can be made symmetric by suitable linear combinations of the equations in (1.2).

In the case of 1–matrix models the operators (1.2) indeed were identified with the positive–frequency part of a Virasoro algebra and generalizations related to higher matrix models were then considered [1]. In this context it is of particular interest to realize this scheme or its generalizations for either linear (Kac-Moody) or non–linear (W_n) algebras. These last ones arise as the relevant algebraic structures for constraints determining the partition function of higher matrix models and strings [3]. The existence of a relationship between realizations of these two categories of algebras [4] leads us to consider here the simpler case of $sl(n)_k$ Kac-Moody algebras represented by first order differential operators.

We first of all consider the problem of constructing "collective" Hamiltonians associated, in the sense defined by (1.1) and (1.2), to $sl(n)_k$ Kac-Moody algebras. We will demonstrate that this can be successfully done. The salient features of this construction, involving a very specific form for the Jacobian J, are established in the general case and illustrated by the examples of $sl(2)$, $sl(3)$ and $sl(4)$ Kac-Moody algebras where these features arise clearly.

We then discuss the problem of deriving W_n-algebra constraints in this framework, restricting ourselves to the construction of the Jacobian itself. We show how a Jacobian obeying W_n-constraints follows from a Jacobian obeying $sl(n)_k$ Kac-Moody constraints by integration over certain well–chosen and suitably weighted variables. This in the present context is the analogue of a Drinfeld-Sokolov reduction [5]. It represents a "Schrödinger picture version" of the BRST mechanism described for example in [6]. The construction is made explicit for $n = 2$ and $n = 3$.

In the conclusion, we give some indications about the possible extensions of the construction, particularly to higher-order differential operators.

2. Hamiltonians Related to Kac-Moody Algebras $sl(n)_k$

The starting point of the construction in the general case is the Wakimoto-Feigin-Frenkel representation of the Kac-Moody algebra $sl(n)_k$ as a set of order-1 differential operators [7]. It can be obtained from the relevant currents in a WZW model when the group–valued field g is adequately parametrized [8]. The Cartan generators take the form:

$$H_a = \sum_{k \in \text{ positive roots}} h_{ak} \frac{\partial}{\partial \gamma_k} + \left(\partial_z \frac{\partial}{\partial \phi_a} + \phi_a \right) \tag{2.1}$$

The generators associated to respectively positive and negative roots take the form:

$$\begin{aligned} J_i^{\pm} = & \sum_{k \in \text{ positive roots}} O_{ik}^{\pm} \frac{\partial}{\partial \gamma_k} + \sum_{a \in \text{ Cartan}} O_{ia}^{\pm} \left(\partial_z \frac{\partial}{\partial \phi_a} + \phi_a \right) \\ & + O_i^{\pm}(\gamma, \partial_\gamma, \cdots) \end{aligned} \tag{2.2}$$

The notation is as follows. The fields ϕ_a indexed by the Cartan generators have an expansion on positive modes only: $\phi_a(z) = \sum_{n \geq 0} \phi_a^n \, z^n$ and their momenta have a conjugate expansion on negative modes only $\frac{\partial}{\partial \phi_a} = \sum_{n \geq 0} \frac{\partial}{\partial \phi_a^n} \cdot z^{-n}$. This gives rise to a mixed field $(\partial_z \frac{\partial}{\partial \phi_a} + \phi_a)$ realizing a $U(1)\,KM$ algebra for each Cartan element. The fields γ_k associated with the positive roots and their conjugate momenta $\frac{\partial}{\partial \gamma_k}$ have an expansion on both positive and negative modes. The coefficients $O_{ik}^{\pm}$ built an invertible matrix, since the number of independent generators must be equal to the number of free parameters in the underlying Lie group; it is known [7] that $h_{ak}, O_{ik}^{\pm}, O_{ia}^{\pm}$ and $O_i^{\pm}$ depend solely on the fields γ_k; generically the choice of suitable γ_k coordinates lead to $O_i^- = 0$.

In order to use the set of KM generators to construct a Hamiltonian on the lines described in (1.1)–(1.3), we must select a maximal closed set of differential generators from (2.1) and (2.2) in order to implement the hermiticity constraints (1.2) on the Hamiltonian. We choose this maximal set to be $J_i^+(z)$ and $H_a^{\geq 0}(z)$, that is, the positive modes in the z–expansion of the Kac-Moody generator $H_a(z)$.

Notice that the choice of $J_i^-(z)$ with $H_a^{\leq 0}(z)$, with the extra property $O_i^-(z) = 0$, would lead to a trivial Jacobian, since $J_i^+(z)$ and $H_a^{\leq 0}$ are pure differentials. It must be emphasized that since $J_i^+(z)$ close a Borel (triangular) subalgebra without central charge, one can at least formally consider any invertible linear combination with γ-dependent coefficients, of the functional differential equations $J_i^+(z) \cdot J = 0$ as an equivalent admissible set of constraints, closing an algebra within the vector space generated by $J_i^+(z)$ with γ-dependent coefficients. Such transformations will be crucial in our derivation. Since only $H_a^{\geq 0}$ is considered, however, it will be practical to restrict oneself to constant linear combinations of the functional differential equations $H_a^{\geq 0} J = 0$. This indeed allows to maintain the global functional notation for the differential operators and the fields.

The derivation runs as follows. One starts with the linear constraints:

$$\begin{gathered} \left\{ \sum_k O_{ik}^+ \frac{\partial}{\partial \gamma_k} + \sum_a O_{ia}^+ \left(\partial_z \frac{\partial}{\partial \phi_a} + \phi_a \right) + O_i^+(\gamma) \right\} J = 0 \\ \left\{ \sum_k \left(h_{ak} \frac{\partial}{\partial \gamma_k} \right)_{n \geq 0} + \phi_a \right\} J = 0 \end{gathered} \tag{2.3}$$

From the invertibility of O_{ik}^+, it follows that there exists a unique linear combination of the first (i-labeled) equations which gives a system of the form:

$$\left\{ \sum_k h_{ak} \frac{\partial}{\partial \gamma_k} + \sum_b q_{ab}(\gamma) (\partial_z \frac{\partial}{\partial \phi_b} + \phi_b) + h_a(\gamma) \right\} J = 0 \tag{2.4}$$

We now make an assumption, not yet proved in the general case but soon to be checked for $n = 2, 3$ and 4:

Assumption A. *The "coupling" matrix $(q_{ab}(\gamma) + \delta_{ab})$ is a γ-independent invertible matrix.*

It follows that the a-labeled equations are now reduced to purely algebraic

equalities.

$$\sum_b \left\{ q_{ab} + \delta_{ab})\phi_b + h_a^{>0}(\gamma_k) \right\} J = 0 \tag{2.5}$$

obtained from projecting (2.4) on positive modes and subtracting from (2.3). The purely algebraic quantity in (2.5) is therefore written as $H_a^{>0}$ minus a linear combination of J^+. From (2.5) it follows that the Jacobian solving (2.3) must be written as:

$$J = \prod_a \delta \left[(q + \mathbf{1})^{-1}_{ba} \, h_a^{>0}(\gamma) + \phi_a \right] \; \tilde{J}(\gamma) \tag{2.6}$$

This ansatz is then plugged back into (2.3). Its consistency is established as follows:

The action of the differential operators (2.3) on a form (2.6) creates terms in $\delta(qh - \phi)$ and $\delta'(qh - \phi)$. These last terms arise uniquely from the action of (2.3) on the functionals $(q + \mathbf{1})^{-1} h^{>0} - \phi$ inside δ. This functional is, as follows from (2.5), a linear combination of generators of the original algebra (2.3) with γ-dependent coefficients of J generators and constant coefficients of H generators. The action of differential operators in (2.3) on a functional is realized as the commutation of the generators in (2.3) with this functional. It gives again a functional of γ, without dependence in ϕ_a since the coefficients of the differentials in (2.3) do not depend on ϕ. Moreover this functional is again a γ-dependent linear combination of the sole J^+ generators, by exact closure (without central charge) of the Borel + (positive modes of) Cartan algebra. A ϕ_a-independent, purely functional linear combination of the generators J^+ necessarily vanishes since the couplings O^+_{ik} to derivatives of J^+_k are invertible. Hence no δ' term arises.

We have thus shown that the operators (2.3) can be consistently applied to functionals of ϕ and γ of the form (2.6). The reduced set of equations obeyed by the functional $\tilde{J}(\gamma)$ reads:

$$\left\{ \sum_{k \in \text{roots}} O^\dagger_{ik} \frac{\partial}{\partial \gamma_k} + \sum_{a \in \text{Cartan}} O^+_{ia} \, \phi_a(\gamma) + O_i(\gamma, \partial_\gamma \cdots) \right\} \tilde{J}(\gamma) = 0 \tag{2.7}$$

Indeed the differential term $\frac{\partial}{\partial \phi_a}$ acts only inside the δ-functional in (2.6) and we

have seen that this action was exactly compensated by the action of the $\frac{\partial}{\partial\gamma}$ terms. ϕ_a is replaced by its value $\phi_a(\gamma)$ extracted from (2.5) and (2.6). The set of operators (2.7) represents the Borel subalgebra (2.3) acting consistently on the subset of functionals (2.6); hence these reduced operators also close the Borel (triangular) subalgebra of the $sl(n)_k$ KM algebra.

The problem now reduces to finding a linear combination of the operators $\tilde{J}_k^+$ in (2.7) such that the kernel $O_{ik}^\dagger$ becomes symmetric, in order to be in the situation described in (1.2). Such a coupling of $\tilde{J}_k^+$ to $\frac{\partial}{\partial\gamma_n}$ can always be achieved since $O_{ik}^\dagger$ is invertible, as:

$$H = \sum_{k,n,l} \frac{\partial}{\partial\gamma_k}(z_1) O_{nl}^\dagger(z_2) S_{nk}(z_2, z_1) \tilde{J}_l^+(z_2) \tag{2.8}$$

with $S_{k\ell}$ an arbitrary symmetric "coupling matrix". Particular forms of $O_{ik}^\dagger$ may lead to more suitable parametrizations of H as we are now going to establish for $n = 2$ and 3.

3. The Example of $sl(2)$ and $sl(3)$

The $sl(2)_k$ KM algebra is represented [7] as:

$$\begin{aligned} J^+ &= -\gamma^2 \frac{\partial}{\partial\gamma} + 2\gamma(\partial_z \frac{\partial}{\partial\phi} + \phi) + k\,\partial\gamma \\ H^0 &= -\gamma \frac{\partial}{\partial\gamma} + (\partial_z \frac{\partial}{\partial\phi} + \phi) \\ J^- &= \frac{\partial}{\partial\gamma} \end{aligned} \tag{3.1}$$

Choosing as closed algebra generators for the Hamiltonian J^+ and $(H^0)^{n>0}$, we consider the combination $(-\frac{1}{\gamma}J^+ + H^0)^{n>0}$, which gives the purely algebraic quantity $(-\phi - k\frac{\partial\gamma}{\gamma})^{n>0}$ with coefficient 1 for ϕ, thereby realizing Assumption A. Hence

one sets the Jacobian to be:

$$J(\phi,\gamma) = \delta\left(\phi + (k\frac{\partial\gamma}{\gamma})^{n>0}\right)\tilde{J}(\gamma) \tag{3.2}$$

where $\tilde{J}(\gamma)$ obeys the (functional) equation:

$$\left(-\gamma^2\frac{\partial}{\partial\gamma} + 2\gamma(-k\frac{\partial\gamma}{\gamma})^{n>0} + k\,\partial\gamma\right)\tilde{J} = 0 \tag{3.3}$$

or more transparently, dividing by $\gamma(z)$:

$$\left(-\gamma\frac{\partial}{\partial\gamma} - k\frac{\partial\gamma}{\gamma}\right)_{n>0}\tilde{J} = 0 \qquad \left(-\gamma\frac{\partial}{\partial\gamma} + k\frac{\partial\gamma}{\gamma}\right)_{n<0}\tilde{J} = 0 \tag{3.4}$$

The new equations cannot be written anymore as a single functional differential equation, owing to the different nature of the reduction in (2.7) for ϕ_a and $\frac{\partial}{\partial\phi_a}$. The generic Hamiltonian associated through (1.1) - (1.2) to (3.4) now takes the form:

$$H = -\int dz_1 dz_2 \gamma(z_1)\Omega(z_2,z_1)\frac{\partial}{\partial\gamma(z_1)}\frac{\partial}{\partial\gamma(z_2)} + \text{linear terms} \tag{3.5}$$

Symmetry of the double derivative kernel can be generically achieved by writing:

$$-\gamma(z_1)\Omega(z_2,z_1) = -\gamma(z_2)\Omega(z_1,z_2) \Rightarrow \qquad \Omega(z_1,z_2) = \gamma(z_1)\,S(z_1,z_2) \tag{3.6}$$

with $S(z_1,z_2)$ any invertible symmetric kernel. H becomes:

$$\begin{aligned} H = -\int dz_1 dz_2 \Big\{ &\gamma(z_1)\gamma(z_2)\,S(z_1,z_2)\frac{\partial}{\partial\gamma(z_1)}\frac{\partial}{\partial\gamma(z_2)} \\ +k&\left(\left(\frac{\partial\gamma}{\gamma}(z_1)\right)_{n>0} - \left(\frac{\partial\gamma}{\gamma}(z_1)\right)_{n<0}\right)\gamma(z_2)\cdot S(z_1,z_2)\frac{\partial}{\partial\gamma(z_2)}\Big\} \end{aligned} \tag{3.7}$$

Note that a very simple choice for $S(z_1,z_2)$ would be $\delta(z_1 - z_2)$. From (3.7) and

(1.3) one then writes the Hermitian Hamiltonians. They have the generic form:

$$
\begin{aligned}
H = - \int dz_1 dz_2 \Big\{ & \frac{\partial}{\partial \gamma(z_1)} \gamma(z_1)\gamma(z_2)\, S(z_1, z_2) \frac{\partial}{\partial \gamma(z_2)} \\
& + \frac{1}{4}\alpha(z_1) S(z_1, z_2) \alpha(z_2) + \frac{1}{2}\frac{\delta\alpha}{\delta\gamma}(z_1) S(z_1, z_2)\gamma(z_2)
\end{aligned}
\tag{3.8}
$$

where $\alpha(z)$ denotes the field $\frac{\partial\gamma}{\gamma}(z)_{n>0} - \frac{\partial\gamma}{\gamma}(z)_{n<0}$. For $S = \delta$, the first two terms in (3.8) reduce to the free quadratic Hamiltonian for the field $\Gamma = e^{\gamma}$, since the sign shift in α does not play any role when evaluating $\int dz \alpha^2(z)$. The term $\frac{\delta\alpha}{\delta\gamma}(z)\gamma(z)$ however adds a non-trivial contribution, not to be expressed in terms of the original field γ.

The case of $sl(3)$ is more interesting and more complicated. The generators of $sl(3)_k$ are parameterized [6, 8] as:

$$
\begin{aligned}
J_1^+ &= -\gamma_1^2 \frac{\partial}{\partial\gamma_1} + \gamma_3 \frac{\partial}{\partial\gamma_2} + i\alpha'\gamma_1 \left(\partial_z \frac{\partial}{\partial\phi_a} + \phi_a\right) + (k+1)\partial\gamma_1 \\
J_2^+ &= (\gamma_1\gamma_2 - \gamma_3) \frac{\partial}{\partial\gamma_1} - \gamma_2^2 \frac{\partial}{\partial\gamma_2} - \gamma_2\gamma_3 \frac{\partial}{\partial\gamma_3} + i\alpha'\gamma_2 \left(\partial_z \frac{\partial}{\partial\phi_b} + \phi_b\right) + k\partial\gamma_2 \\
J_3^+ &= \gamma_1(\gamma_1\gamma_2 - \gamma_3) \frac{\partial}{\partial\gamma_1} - \gamma_2\gamma_3 \frac{\partial}{\partial\gamma_2} - \gamma_3^2 \frac{\partial}{\partial\gamma_3} + i\alpha'(\gamma_3 - \gamma_1\gamma_2) \left(\partial_z \frac{\partial}{\partial\phi_a} + \phi_a\right) \\
&\quad + i\alpha'\gamma_3(\partial_z \frac{\partial}{\partial\phi_b} + \phi_b) - (k+1)\partial\gamma_1\gamma_2 + k\partial\gamma_3 \\
H_a &= 2\gamma_1 \frac{\partial}{\partial\gamma_1} - \gamma_2 \frac{\partial}{\partial\gamma_2} + \gamma_3 \frac{\partial}{\partial\gamma_3} - \alpha' \left(\phi_a + \partial_z \frac{\partial}{\partial\phi_a}\right) \\
H_b &= -\gamma_1 \frac{\partial}{\partial\gamma_1} + 2\gamma_2 \frac{\partial}{\partial\gamma_2} + \gamma_3 \frac{\partial}{\partial\gamma_3} - \alpha' \left(\phi_b + \partial_z \frac{\partial}{\partial\phi_b}\right) \\
J_1^- &= \frac{\partial}{\partial\gamma_1} + \gamma_2 \frac{\partial}{\partial\gamma_3} \qquad J_2^- = \frac{\partial}{\partial\gamma_2} \qquad J_3^- = \frac{\partial}{\partial\gamma_2} \qquad \alpha' = \sqrt{2k+6}
\end{aligned}
\tag{3.9}
$$

Choosing as a closed set of differential operators the triangular upper Borel subalgebra J_i^+ plus the positive modes of the Cartan algebra $H_a^{n>0}, H_b^{n>0}$, the elimination

of $\frac{\partial}{\partial\gamma}$ terms following the general pattern (2.4),(2.5) is achieved by:

$$\tilde{H}_a = \frac{\gamma_2}{\gamma_3(\gamma_1\gamma_2-\gamma_3)}\left\{(\gamma_3-\gamma_1\gamma_2)J_1^+ + \frac{\gamma_1\gamma_3}{\gamma_2}\,J_2^+ - \gamma_1\,J_3^+\right\} - \frac{1}{3}(H_a-H_b)$$
$$\tilde{H}_b = \frac{\gamma_2}{\gamma_3(\gamma_1\gamma_2-\gamma_3)}\left\{(\gamma_3-\gamma_1\gamma_2)J_1^+ - \frac{\gamma_1\gamma_3}{\gamma_2}\,J_2^+ + (2\frac{\gamma_3}{\gamma_2}-\gamma_1)J_3^+\right\} - (H_a+H_b) \tag{3.10}$$

and the ϕ-terms are indeed coupled to a constant matrix, thereby confirming Assumption A:

$$\tilde{H}_a^{n>0} = (\phi_a-\phi_b) + 3\left(\frac{\gamma_1\gamma_2}{\gamma_1\gamma_2-\gamma_3}\left\{(k+1)\frac{\partial\gamma_1}{\gamma_1} + k\frac{\partial\gamma_2}{\gamma_2} - k\frac{\partial\gamma_3}{\gamma_3}\right\}\right)^{n>0}$$
$$\tilde{H}_b^{n>0} = (-\phi_a-\phi_b) - \left(\frac{\gamma_1\gamma_2}{\gamma_1\gamma_2-\gamma_3}\left\{(k+1)\frac{\partial\gamma_1}{\gamma_1} + k\frac{\partial\gamma_2}{\gamma_2} - k\frac{\partial\gamma_3}{\partial\gamma_3}\right\} + 2k\frac{\partial\gamma_3}{\gamma_3}\right)^{n>0} \tag{3.11}$$

which allows us to consistently eliminate the ϕ dependence in J, following (2.6):

$$J(\phi_a,\phi_b,\gamma) = \delta(\tilde{H}_a^{n>0})\delta(\tilde{H}_b^{n>0})\,\tilde{J}(\gamma_1,\gamma_2,\gamma_3) \tag{3.12}$$

One checks by an immediate computation that $J^+_{1,2,3}$ indeed annihilate the functionals $\tilde{H}_a, \tilde{H}_b$. The equations for the Jacobian are reduced to:

$$\left\{-\gamma_1^2\frac{\partial}{\partial\gamma_1} + \gamma_3\frac{\partial}{\partial\gamma_2} + i\alpha'\gamma_1\left(\phi_a(\gamma_1,\gamma_2,\gamma_3)\right) + (k+1)\partial\gamma_1\right\}\tilde{J} = 0$$
$$\left\{(\gamma_1\gamma_2-\gamma_3)\frac{\partial}{\partial\gamma_1} - \gamma_2^2\frac{\partial}{\partial\gamma_2} - \gamma_2\gamma_3\frac{\partial}{\partial\gamma_3} + i\alpha'\gamma_2\left(\phi_b(\gamma_1,\gamma_2,\gamma_3)\right) + k\partial\gamma_2\right\}\tilde{J} = 0$$
$$\Big\{\gamma_1(\gamma_1\gamma_2-\gamma_3)\frac{\partial}{\partial\gamma_1} - \gamma_2\gamma_3\frac{\partial}{\partial\gamma_2} - \gamma_3^2\frac{\partial}{\partial\gamma_3} + i\alpha'\gamma_3\left(\phi_b(\gamma_1,\gamma_2,\gamma_3)\right)$$
$$+ i\alpha'(\gamma_3-\gamma_1\gamma_2)\phi_a(\gamma_1,\gamma_2,\gamma_3) + k\partial\gamma_3 - (k+1)\partial\gamma_1\gamma_2\Big\}\,\tilde{J} = 0 \tag{3.13}$$

where ϕ_a, ϕ_b are expressed in terms of positive modes of the functionals of $\gamma_1, \gamma_2, \gamma_3$ obtained in (3.11). The construction of the Hamiltonians associated to the set of closed differential equations (3.13) follows from (2.8) as $H = \sum_{k,n,l}\frac{\partial}{\partial\gamma_k}(z_1)O^\dagger_{nl}(z_2)S_{nk}(z_2,z_1)\tilde{J}^+_l(z_2)$ We recall that S is any symmetric invertible kernel – the simplest example being $S_{ij}(z_1,z_2) = \delta_{ij}\,\delta(z_1-z_2)$.

This construction is noticeably simplified if one considers the equivalent linear system to (3.13) obtained by diagonalizing the coefficient matrix of the derivatives. One gets a set of equations closely resembling (3.4):

$$
\begin{aligned}
&\{\gamma_1\frac{\partial}{\partial\gamma_1}+\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2-\gamma_3)\{\phi_a-\frac{<\phi_a>}{2}-k\frac{\partial\gamma_2}{\gamma_2}+\frac{k}{2}\frac{\partial\gamma_3}{\gamma_3}\}\}\tilde{J}=0\\
&\{\gamma_2\frac{\partial}{\partial\gamma_2}+\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2-\gamma_3)\{\phi_a-\frac{<\phi_a>}{2}-(k+1)\frac{\partial\gamma_1}{\gamma_1}-\frac{k}{2}\frac{\partial\gamma_3}{\gamma_3}\}\}\tilde{J}=0\\
&\Big\{2\gamma_3\frac{\partial}{\partial\gamma_3}+\gamma_3^2(\gamma_1\gamma_2-\gamma_3)(\gamma_1\gamma_2-2\gamma_3)\{\phi_a-\frac{<\phi_a>}{2}+(\phi_b-\frac{<\phi_b>}{2})\}\\
&+\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2-\gamma_3)\{\phi_a-\frac{<\phi_a>}{2}-(\phi_b-\frac{<\phi_b>}{2})-k\frac{\partial\gamma_2}{\gamma_2}+(k+1)\frac{\partial\gamma_1}{\gamma_1}\}\Big\}\tilde{J}=0
\end{aligned}
\tag{3.14}
$$

where $<\phi_{a,b}>$ denoted the expression of $\phi_{a,b}$ in terms of the field γ obtained from (3.10) *without* projecting out the negative modes:

$$
\begin{aligned}
<\phi_a>&=-2\frac{\gamma_1\gamma_2}{\gamma_1\gamma_2-\gamma_3}\left\{(k+1)\frac{\partial\gamma_1}{\gamma_1}+k\frac{\partial\gamma_2}{\gamma_2}-k\frac{\partial\gamma_3}{\gamma_3}\right\}-k\frac{\partial\gamma_3}{\gamma_3}\\
<\phi_b>&=\frac{\gamma_1\gamma_2}{\gamma_1\gamma_2-\gamma_3}\left\{(k+1)\frac{\partial\gamma_1}{\gamma_1}+k\frac{\partial\gamma_2}{\gamma_2}-k\frac{\partial\gamma_3}{\gamma_3}\right\}-k\frac{\partial\gamma_3}{\gamma_3}
\end{aligned}
\tag{3.15}
$$

Hence $\phi_a - 1/2 <\phi_a>$ is exactly the difference between positive and negative modes of $<\phi_a>$, a combination which already occurred in the case of $sl(2)$, see (3.7).

The general $sl(3)$ Hamiltonian thus follows from (3.15) and (2.8), and the explicit Hermitian Hamiltonian from (1.3). It will exhibit the same features as in the $sl(2)$ case: a free quadratic part in the fields γ and its conjugate $\frac{\partial}{\partial\gamma}$, and a non-trivial contribution from the γ functional derivative of the split field $<\phi_{a,b}>_{n<0}$ $-<\phi_{a,b}>_{n>0}$.

Using the formulae given in [8] we have also proved the validity of Assumption A for the algebra $sl(4)_k$. We do not include the explicit proof here due to its lengthiness. It follows from long but straightforward computations on similar lines as (3.9)-(3.11).

The construction in Section 2 is now safely established for $sl(2,3,4)$. Moreover the non-triviality of the $sl(3,4)$ cases and the simplicity of the actual linear combinations required to prove Assumption A leads us to expect that it can be proved on general grounds for all KM algebra representations of the type (2.1),(2.2), thereby validating the whole scheme at one strike.

4. From Kac-Moody to W-algebras. The Jacobian

Once one has obtained a Jacobian associated with a set of Kac-Moody generators, as above for $sl(2)$ and $sl(3)$, one can extract from it a "Jacobian" associated with a different, non-linear algebra of higher-order differential operators which can be shown in the cases of $sl(2)$ and $sl(3)$ to realize the W_2 (Virasoro) and W_3 algebra (more exactly a centerless sector of these algebras). The general procedure is as follows.

One goes back to the full Jacobian (2.5) obeying the complete set of $sl(n)_k$ KM-operators equations. One then implements on this Jacobian the *classical* reduction leading from the KM to a W_n–algebra. Classically this is achieved by setting a particular subset of momenta $\beta_i = \frac{\partial}{\partial\gamma_i}$ to constant values 0 or 1, the particular choice corresponding to particular W_n algebras [4]. This will correspond in our formalism to integrating out a subset of variables, thereby deriving an effective Jacobian depending on the reduced set. Specifically we shall integrate with respect to γ–variables all equations of the KM set, adding a pre-factor $e^{\gamma_i^0}$ each time one classical momentum β_i should be put to 1. In this way, the derivatives $\frac{\partial}{\partial\gamma_i}$ are formally put to their reduced values 0 or 1. The result is generically a set of coupled differential equations for momenta of J. The assumption we make, proved here for $sl(2)_k$ and $sl(3)_k$, is that this set of integral differential equations can be transformed into a set of purely differential equations for the integrated density $\int d\gamma_1 \cdots d\gamma_n e^{\gamma_i^0} \cdots J(\phi_a \cdots \gamma_1 \cdots \gamma_n)$. Since the equations one originally gets are equations relating various momenta of this density, (i.e. $\int d\gamma_1 \cdots d\gamma_n \, \gamma_i^{p_1} \cdots \gamma_n^{p_n} \, e^{-} \, J(\gamma_1 \cdots \gamma_n, \phi_a, \phi_n))$ this means that we conjecture the

existence of (rank $sl(n)$) combinations of the differential operators (J_i^+, H_a^+) where *all* higher momenta of J are eliminated.

a) **$\underline{sl(2)\text{ case}}$**

The Jacobian obeys the two sets of equations:

$$\left(-\gamma^2\frac{\partial}{\partial\gamma}+2\gamma(\phi+\partial_z\frac{\partial}{\partial\phi})+k\partial\gamma\right)J=0 \tag{4.1}$$

$$\left(-\gamma\frac{\partial}{\partial\gamma}+\partial\varphi\right)_{n\geq 0}J=0 \tag{4.2}$$

From (4.1), integrating over γ after multiplying by e^{γ_0}, one gets

$$\int e^{-\gamma_0}(-\gamma^2+2\gamma A+2\gamma\,\partial\varphi+\partial\gamma)J=0 \tag{4.3}$$

A is a regularized constant, formally equal to $\sum_{-\infty}^{+\infty}1$ and arising from partial integration of a term $\gamma\frac{\partial}{\partial\gamma}$. From now on $\partial\varphi$ will be understood as a short notation for $(\phi+\partial_z\frac{\partial}{\partial\phi})$.It denotes a field of conformal spin 1. From (4.3) one gets first,

$$\int e^{-\gamma_0}(-\gamma+A+\partial\varphi)_{n\geq 0}J=0 \tag{4.4}$$

and multiplying (4.3) by $(-\gamma+\partial\varphi)_p$ for $p\in\mathbf{Z}$.

$$\forall p,\forall n\geq 0\int e^{-\gamma_0}\left((-\gamma+\partial\varphi)_p(-\gamma+\partial\varphi)_n+(-\gamma+\partial\varphi)_p\delta_{n,0}A-\gamma_{n+p}\right)J=0 \tag{4.5}$$

Summing now (4.5) for all values of $p,n\geq 0$ such that $n+p=m\geq 0$ gives

$$\int e^{-\gamma_0}\left((-\gamma+\partial\varphi)^2_m-(A-m)\gamma_m-A^2\delta_{m,0}\right)J=0 \tag{4.6}$$

Indeed, in order to get $(-\gamma+\partial\varphi)^2_m$ as a result of the summation, one needs to sum for n positive from $\frac{m}{2}+1$ to ∞ twice and $n=\frac{m}{2}$ once. This explains the $(A-m)$ factor in front of γ_m as $\sum_{-\infty}^{+\infty}1(=A)-2\sum_{-m/2}^{m/2}1(=m)$.

The $m\gamma_m$ term turns into $\partial\gamma$ which will be replaced by $\partial^2\varphi$ using (4.4) and therefore will contribute to the final value of the central charge.

Putting together now (4.6), (4.4) and (4.3) results in a complete elimination of the higher momenta of J, leaving only:

$$\int e^{\gamma_0} \left((\partial\varphi + A/2)^2 - A^2/4 + (k+1)\partial^2\varphi\right)_{m\geq 0} J = 0 \qquad (4.7)$$

from which the weighted density J is immediately seen to obey a set of Virasoro constraints $\sim [(\partial\varphi)^2 + (k+1)\partial^2\varphi]^+$ without a central term.

sl(3) case.

This case is more interesting and complicated. One can however check that there exist a suitable combination of generators eliminating the highest momenta. Indeed one has (setting the conjugate fields $\beta_{1,2} = \frac{\partial}{\partial\gamma_{1,2}}$ to 1 and $\beta_3 = \frac{\partial}{\partial\gamma_3}$ to 0 by using the weighted integration measure $d\gamma_1 d\gamma_3 d\gamma_3 e^{\gamma_1^0} e^{\gamma_2^0}$)

$$\begin{aligned}&(2H_a + H_b)(2H_b + H_a)(H_a - H_b) - (J_3 - 27(H_a + H_b)J_2 + 9(H_a + 2H_b)(J_1 + J_2))\\ &= (2\partial\varphi_a + \partial\varphi_b)(2\partial\varphi_b + \partial\varphi_a)(\partial\varphi_a - \partial\varphi_b) + 9(\partial\varphi_a + 2\partial\varphi_b)\partial\gamma_1\end{aligned} \qquad (4.8)$$

It follows that one should consider, following a similar scheme as in (4.5), the realization of the positive modes of this generator by taking suitable momenta of the equations $(H_a - H_b)^{\geq 0} J = 0, (2H_a + H_b)^{\geq 0} J = 0, (2H_b + H_a)^{\geq 0} J = 0$. Such a realization, as we have seen in (4.6), modifies the non-derivative terms $k\partial\gamma_i$ by constant shifts of k. Extra terms due to part integration also arise. The statement now is that the leading order of relevant momenta of the density $e^{\gamma_0} J$ is eliminated by this suitable combination of generators. We conjecture that this statement be true at lower orders – the extra term has exactly the form $F(\phi_a)\partial\gamma_i$ which the part-integration and index-shift effects generate – in which case the integrated density is annihilated by the positive modes of a W_3 current:

$$W = (2\partial\varphi_a + \partial\varphi_b)(2\partial\varphi_b + \partial\varphi_a)(\partial\varphi_a - \partial\varphi_b) \qquad (4.9)$$

(4.9) is indeed the canonical realization of a W_3 current.

The Virasoro algebra part is obtained as follows: The combination of generators $H_a^2 + H_b^2 + H_a H_b + 3(J_1 + J_2)$ yields :

$$L = (\partial\varphi_a)^2 + (\partial\varphi_b)^2 + \partial\varphi_a\partial\varphi_b + 3(k+1)\partial\gamma_1 + 3\partial\gamma_2\,. \tag{4.10}$$

It follows that up to lower-order terms in γ, this combination will allow to eliminate all γ–momenta of $\int e^{\gamma_0} J$. But lower-order here means only linear, and linear momenta in γ can be re-expressed as linear momenta in $(\phi + \partial_z \frac{\partial}{\partial\phi})_{a,b}$ from H-equations. Hence the complete elimination of γ-momenta is proved to be achieved for the Virasoro generators.

We would like to conclude this section by reminding that the Jacobian J is necessarily of the form $J = \delta(\phi - \phi(\gamma))\tilde{J}(\gamma)$. Hence integration of J with respect to γ will yield very non-trivial dependences on the remaining Cartan algebra parameters ϕ_a.

5. Conclusions and Extensions

Let us briefly summarize the scheme and the construction presented in this paper. We have described a construction of general $sl(n)_k$-based collective hamiltonians. To this effect we have postulated the existence of a consistent construction by which, choosing a Borel subalgebra of a Kac-Moody algebra $sl(n, \mathbf{C})$ generated by the set $\{J_a, H\}$ one obtains a corresponding Jacobian obeying (1.2), expressed as $J = \delta(\phi_a - \phi_a(\gamma))\tilde{J}(\gamma)$. The algebra of constraints obeyed by this Jacobian was then reformulated in two ways:

1) a reduced linear strictly triangular subalgebra, obtained by eliminating trivially the ϕ_a variables following (2.6) : $J = \tilde{J}(\gamma), \phi_a = \phi_a(\gamma), \frac{\partial}{\partial\phi_a}$ drops out.

2) a nonlinear W_n algebra (positive-index part) obtained by eliminating the γ variables by integration, leading to $J = J(\phi_a)$.

We have then explicitely demonstrated the consistency of the scheme for $sl(2)_k$, $sl(3)_k$ and partially for $sl(4)_k$.

One obvious generalization to be achieved now is the extension of this proof to all $sl(n)_k$ by proving the global validity of our assumptions, and the subsequent construction of the $sl(n)_k$-Hamiltonians by the already well-established scheme leading to (3.5) and (3.13). This should follow from the WZW represntation of KM algebras derived in [8].

These theories will be useful as generalized field theories of matrix models and strings. It was seen in [1] that through stochastic quantization matrix models lead to generalized (loop space) Hamiltonians, the Jacobians being identified with the partition function. Based on the constraints obeyed by the present Jacobians and following procedure 2) given above we see the emergence of the W_n algebra of equations constraining the partition function of n-matrix models.

Finally it is relevant to state how the scheme (1.1)-(1.2) can be extended to Hamiltonians involving differential operators of higher order. The general statement reads:

For any differential operator of order n:

$$H = \sum_{k=0}^{n} O^{(k)}_{i_1 \ldots i_n}(\gamma) \prod_{r=1}^{k} \frac{\partial}{\partial \gamma_{i_r}}$$

the even-order (resp. odd–order) terms can be eliminated and the operator made antihermitian (resp. hermitian) for n odd (resp. even) provided a solution exists to the set of equations:

$$\forall p = 1 \ldots n/2,$$

$$([\prod_{k=1}^{p} \frac{\partial}{\partial \gamma_{i_k}}] O^{(n-p+1)}_{i_1 \ldots i_{n-p+1}} - [\prod_{k=1}^{p-1} \frac{\partial}{\partial \gamma_{i_k}}] O^{(n-p)}_{i_2 \ldots i_{n-p+1}} + \ldots + O^{(n-2p+1)}_{i_{p+1} \ldots i_{n-p+1}}) J = 0 \qquad (5.1)$$

and provided the odd-order (resp. even-order) coefficients $O^{(k)}$ are totally symmetric under permutation of their indices. The solution to this system is the looked-for

Jacobian.

This extension should allow a direct study, at the Hamiltonian level, of the case of W_n algebras, represented as they are by algebras of n-th order differential operators, to be identified from (5.1). This will be adressed in a forthcoming study.

Acknowledgements We wish to thank the referee for fruitful suggestions. J. A. thanks all members of Brown University Physics Department for their kind hospitality.

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